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Tugas Latihan 4.5 Anvek (C)

12). Misalkan $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$ dan $r = |\mathbf{r}|$, periksalah kebenaran persamaan berikut ini!

a). $\nabla \cdot \mathbf{r} = 3$

$$\nabla \cdot \mathbf{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= 1 + 1 + 1$$

$$= 3 \quad \text{[TERBUKTI]}$$

b). $\nabla^2 r^3 = 12r$

$$\nabla(\nabla r^3) = \nabla\left[\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\sqrt{x^2+y^2+z^2}\right)^3\right]$$

$$= \nabla\left[\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(x^2+y^2+z^2\right)^{3/2}\right]$$

$$= \nabla\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}.2x\,\hat{i} + \frac{3}{2}(x^2+y^2+z^2)^{1/2}2y\,\hat{j} + \frac{3}{2}(x^2+y^2+z^2)^{1/2}.2z\,\hat{k}\right]$$

$$= \nabla\left[3x(x^2+y^2+z^2)^{1/2}\hat{i} + 3y(x^2+y^2+z^2)^{1/2}\hat{j} + 3z(x^2+y^2+z^2)^{1/2}\hat{k}\right]$$

$$= \nabla\left[\left(3(x^2+y^2+z^2)^{1/2}\right)\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right]$$

$$= \left[\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(3(x^2+y^2+z^2)^{1/2}(x\hat{i} + y\hat{j} + z\hat{k})\right)\right]$$

$$= \left[3\left((x^2+y^2+z^2)^{1/2} + x.\frac{1}{2}(x^2+y^2+z^2)^{-1/2}.2x\right)\right] + \left[3\left((x^2+y^2+z^2)^{1/2} + \frac{y}{2}(x^2+y^2+z^2)^{-1/2}.2y\right)\right]$$

$$+ \left[3\left((x^2+y^2+z^2)^{1/2} + \frac{z}{2}(x^2+y^2+z^2)^{-1/2}.2z\right)\right]$$

$$= 3(x^2+y^2+z^2)^{1/2} + 3x^2(x^2+y^2+z^2)^{-1/2}$$
$$3(x^2+y^2+z^2)^{1/2} + 3y^2(x^2+y^2+z^2)^{-1/2}$$
$$3(x^2+y^2+z^2)^{1/2} + 3z^2(x^2+y^2+z^2)^{-1/2}$$

$$= 9r + \left[\frac{3}{r}\left[x^2+y^2+z^2\right]\right]$$

$$= 9r + \left[\frac{3r^2}{r}\right]$$

$$= 9r + 3r$$

$$= 12r \quad \text{[TERBUKTI]}$$

c). $\nabla \cdot r\mathbf{r} = 4r$

$$※\ r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$$

$$\nabla \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\left(\sqrt{x^2+y^2+z^2}\right) = \left[\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\left(\sqrt{x^2+y^2+z^2}\right)\right)\right]$$

$$\hookrightarrow = \left[(x^2+y^2+z^2)^{1/2} + x.\frac{1}{2}.(x^2+y^2+z^2)^{-1/2}.2x + (x^2+y^2+z^2)^{1/2} + \frac{y}{2}(x^2+y^2+z^2)^{-1/2}.2y + (x^2+y^2+z^2)^{1/2} + \frac{z}{2}(x^2+y^2+z^2)^{-1/2}.2z\right]$$

$$= 3r + \frac{1}{r}(x^2+y^2+z^2)$$

$$= 3r + \frac{r^2}{r}$$

$$= 3r + r$$

$$= 4r \quad [\text{TERBUKTI}] \; //$$

d). $\nabla r = \mathbf{r}/r$

$$\nabla r = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left[\sqrt{x^2+y^2+z^2}\right]$$

$$= \frac{2x}{\sqrt{x^2+y^2+z^2}}\cdot\frac{1}{2}\hat{i} + \frac{1}{2}\cdot\frac{2y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{1}{2}\cdot\frac{2z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$= \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{\mathbf{r}}{r} \; // \quad [\text{TERBUKTI}]$$

e). $\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right]$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left[(x^2+y^2+z^2)^{-1/2}\right]$$

$$= -\frac{1}{2}\cdot\frac{2x\,\hat{i}}{(\sqrt{x^2+y^2+z^2})^3} - \frac{1}{2}\cdot\frac{2y\,\hat{j}}{(\sqrt{x^2+y^2+z^2})^3} - \frac{1}{2}\cdot\frac{2z\,\hat{k}}{(\sqrt{x^2+y^2+z^2})^3}$$

$$= \frac{-(x\hat{i}+y\hat{j}+z\hat{k})}{(\sqrt{x^2+y^2+z^2})^3}$$

$$= -\frac{\mathbf{r}}{r^3} \quad [\text{TERBUKTI}] \; //$$

f). $\nabla \times \mathbf{r} = 0$

$$\nabla \times \mathbf{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \times \left(x\hat{i} + y\hat{j} + z\hat{k} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$= 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$$

$$= \mathbf{0} \quad /\!/$$ [Terbukti]

g). $\nabla \ln r = \mathbf{r}/r^2$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \times \left(\ln\sqrt{x^2+y^2+z^2} \right)$$

$$= \frac{1}{(x^2+y^2+z^2)^{1/2}} \cdot \frac{1}{2} \cdot \frac{2x}{(x^2+y^2+z^2)^{1/2}}\hat{i} + \frac{1}{(x^2+y^2+z^2)} \cdot \frac{1}{2} \cdot \frac{2y}{(x^2+y^2+z^2)^{1/2}}\hat{j} +$$

$$\frac{1}{(x^2+y^2+z^2)^{1/2}} \cdot \frac{1}{2} \cdot \frac{2z}{(x^2+y^2+z^2)^{1/2}}\hat{k}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}$$

$$= \frac{\mathbf{r}}{r^2} \quad /\!/$$ [TERBUKTI]

h). $\nabla \mathbf{r} f(r) = 3f(r) + |\mathbf{r}| \dfrac{df}{dr}$

$$= \nabla \mathbf{r} \cdot f(r) + \mathbf{r} \cdot \Delta f(r)$$

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$$= 3 \cdot f(r) + (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)^{1/2}} \frac{df}{dr}$$

$$= 3 \cdot f(r) + (x^2+y^2+z^2)^{1/2} \frac{df}{dr}$$

$$= 3f(r) + |\mathbf{r}| \frac{df}{dr}$$ (TERBUKTI)

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13) a). Buktikan bahwa div (grad f) $= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$

$$\text{div}(\text{grad } f) = \text{div}(\nabla f) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(f(x,y,z)\right)$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial f}\hat{k}\right)$$

$$\nabla(\nabla f) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{\partial f_1}{\partial x}\hat{i} + \frac{\partial f_2}{\partial y}\hat{j} + \frac{\partial f_3}{\partial z}\hat{k}\right)$$

$$= \left[\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}\right]$$

$$= \nabla^2 f \quad \text{[TERBUKTI]}$$

b). Jika $\phi = x^2 z - 3xy^2 z - xy^2$, maka tentukan $\nabla\phi$, $|\nabla\phi|$, dan laplace ϕ pada $(1,1,0)$.

• $\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(x^2 z - 3xy^2 z - xy^2\right)$

$$= (2xz - 3y^2 z - y^2)\hat{i} + (0 - 6xyz - 2xy)\hat{j} + (x^2 - 3xy^2 - 0)\hat{k}$$

Pada titik $(1,1,0)$

$$= (0 - 0 - 1)\hat{i} + (-0 - 2)\hat{j} + (1 - 3)\hat{k}$$

$$= -\hat{i} - 2\hat{j} - 2\hat{k}$$

• $|\nabla\phi| = \sqrt{(-1)^2 + (-2)^2 + (-2)^2}$

$$= \sqrt{1 + 4 + 4}$$

$$= \sqrt{9}$$

$$= 3$$

• $\nabla^2 f = \left(\frac{\partial^2 f}{\partial x^2}\hat{i} + \frac{\partial^2 f}{\partial y^2}\hat{j} + \frac{\partial^2 f}{\partial z^2}\hat{k}\right)\left(x^2 z - 3xy^2 z - xy^2\right)$

$$= \left((2z) + (6xz - 2x) + (0)\right)$$

$$= (2z + 6xz - 2x)$$

Pada titik $(1,1,0)$

$$\nabla^2 f \Big|_{(1,1,0)} = (2(0) + 6(1)(0) - 2(1))$$

$$= -2$$

14). Jika $\mathbf{F} = \mathbf{r}/r^p$ carilah div $\mathbf{F}$. Apakah terdapat nilai p hingga berlaku div $\mathbf{F} = 0$

Jawab : misalkan $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$, maka $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$

$$\nabla \mathbf{F} = \nabla \mathbf{r} . r^{-p}$$
$$= \mathbf{r} . \nabla r^{-p} + \nabla \mathbf{r} . r^{-p}$$

※ $$\nabla r^{-p} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)^{-p/2}$$
$$= \left[-\frac{p}{2}(x^2+y^2+z^2)^{\frac{-p-2}{2}} . 2x\,\hat{i} + \left(-\frac{p}{2}(x^2+y^2+z^2)^{\frac{-p-2}{2}} 2y\right)\hat{j} + \left(-\frac{p}{2}(x^2+y^2+z^2)^{\frac{-p-2}{2}} . 2z\right)\hat{k}\right]$$
$$= -p\,(x^2+y^2+z^2)^{\frac{-p-2}{2}} . \left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$
$$= -p . r^{-p-2} . \mathbf{r}$$

※ $$\nabla \mathbf{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$
$$= 1+1+1$$
$$= 3$$

$$\Rightarrow \nabla \mathbf{F} = \mathbf{r} . \nabla r^{-p} + \nabla \mathbf{r} . r^{-p}$$
$$= \mathbf{r} . \left(-p . r^{-p-2} . \mathbf{r}\right) + 3r^{-p}$$
$$= \mathbf{r} . \mathbf{r}\,(-p) . r^{-p-2} + 3r^{-p}$$
$$= r^2 . r^{-p-2}\,(-p) + 3r^{-p}$$
$$= r^{-p}(-p) + 3r^{-p}$$
$$= r^{-p}\left[3-p\right]$$

Pada soal diketahui bahwa $\nabla \mathbf{F} = 0$, maka

$$r^{-p}\left[3-p\right] = 0$$

$$r^{-p} = 0 \text{ (tak ada) } \vee \; 3-p = 0$$
$$p = 3$$

Jadi, nilai p yang memenuhi adalah 3

JOYKO

15). Dapatkan derivatif berarah dari $\varphi = 4xz^2 - 3x^2y^2z$ pada $(2,-1,2)$ dalam arah $2\hat{i} - 3\hat{j} + 6\hat{k}$

$$\nabla\varphi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(4xz^2 - 3x^2y^2z)$$

$$= \left((4z^2 - 6xy^2z)\hat{i} + (0 - 6x^2yz)\hat{j} + (8xz - 3x^2y^2)\hat{k}\right)$$

$$\nabla\varphi\Big|_{(2,-1,2)} = \left((16-24)\hat{i} + (48)\hat{j} + (32-12)\hat{k}\right)$$

$$= \left(-8\hat{i} + 48\hat{j} + 20\hat{k}\right)$$

Vektor satuan dari $\hat{n} = 2\hat{i} - 3\hat{j} + 6\hat{k}$, anggap $\vec{a}$

$$\vec{a} = \frac{(2\hat{i} - 3\hat{j} + 6\hat{k})}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}}$$

$$= \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{49}}$$

$$= \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

Maka, diperoleh turunan berarah

$$\nabla\varphi.\vec{a} = (-8\hat{i} + 48\hat{j} + 20\hat{k}).\left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$$

$$= -\frac{16}{7} - \frac{144}{7} + \frac{120}{7}$$

$$= -\frac{160}{7} + \frac{120}{7}$$

$$= -\frac{40}{7}$$

$$= -5\frac{5}{7}$$

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16). Suatu benda mempunyai massa m, berputar dalam suatu orbit melingkar dengan kecepatan sudut ω akan mengalami gaya sentrifugal yang diberikan oleh
$\mathbf{F}(x,y,z) = m\omega^2 \mathbf{r}$.

Tunjukkan bahwa $f(x,y,z) = \frac{1}{2} m\omega^2 (x^2+y^2+z^2)$ adalah fungsi potensial untuk $\mathbf{F}$

Persamaan bola : $r^2 = x^2+y^2+z^2$

$$f(x,y,z) = \int m.\omega^2 r \, dr$$

$$= \frac{1}{2} m\omega^2 r^2 + C$$

$$= \frac{1}{2} m\omega^2 (x^2+y^2+z^2) + C$$

Saat c = 0 , maka :

$$f(x,y,z) = \frac{1}{2} m\omega^2 (x^2+y^2+z^2)$$